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On the deformation of almost cylindrical elastic beams

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Abstract

Background: The Saint-Venant problem for porous elastic cylinders is of interest both from the technical and mathematical point of view. The intended applications of solution are to mechanics of bone and to some engineering structures.

Method: This work investigates the Saint-Venant problem for almost prismatic bars made of an isotropic porous material. We express the solution in terms of the solutions of two problems concerning the deformation of a right cylinder.

Results and Conclusion: We use the method to study the extension of an almost prismatic conical frustum. It is found that the displacement vector is a polynomial of two degree in the cartesian coordinates. The volume fraction field depends linearly on the axial coordinate. The solution contains terms characterizing the influence of the material porosity and the dependence on the lateral surface.

Keywords: Elastic bars, Porous bodies, Almost cylindrical beams, Elastic materials with voids

Background

A generalization of the classical theory of elasticity is the theory of elastic materials with voids established by Nunziato and Cowin (1979) and Cowin and Nunziato (1983). The intended application of the theory is to elastic bodies with pores which are distributed throughout the material. In the framework of the linear theory of isotropic elastic materials with voids, the deformation of the right cylinders has been the subject of various investigations. Cowin and Nunziato (1983) have studied the pure bending of a cylinder made of a homogeneous material. The problem of extension and bending for nonhomogeneous porous elastic bodies has been investigated by Ciarletta and Iesan (1993), Ieşan and Nappa (1994), Ieşan and Scalia (2007), and Ieşan and Scalia (2009). A treatment of Saint Venant's problem for homogenous and isotropic porous elastic cylinders has been presented by Dell'Isola and Batra (1997), Ieşan and Quintanilla (1995), Ieşan (2009), and Ieşan (2011).

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In the context of the classical elasticity, the mechanical behavior of the noncylindrical elastic bars has been studied in many papers (see, e.g., Dryden (2007), Zupan and Saje (2006), You et al. (2002), and the references therein). These bodies are of interest both from the technical and mathematical point of view. The present paper is concerned with the Saint-Venant problem for almost cylindrical bars made of porous elastic materials. In the classical elastostatics, the deformation of almost cylindrical bars has been studied in various papers (see, e.g., Bors (1973), Chirita (1983), Khatiashvili (1983b), Khatiashvili (1983a), and the references therein). First, we present the basic equations of the linear theory of isotropic porous elastic solids and the formulation of the problem. The next section is devoted to the solution of the Saint-Venant problem for almost cylindrical bars. The problem is reduced to the solving of a problem of Almansi type and to the Saint-Venant problem for a right cylinder. In the next section we use the method to solve the problem of extension of a conical frustum. The solution is expressed in terms of solutions of some problems associated with the deformation of a right circular cylinder.

Formulation of the problem

In this paper, we consider the equilibrium theory of elastic materials with voids. A rectangular cartesian coordinate system $Ox_k (k = 1, 2, 3)$ is used. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers $(1, 2)$ whereas Latin subscripts to the range $(1, 2, 3)$; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. We denote by B the interior of a noncilindrical bar of length l, with the ends located at $x_3 = 0$ and $x_3 = l$, and with the lateral surface Π defined by

$$F\left[x_1(1-kx_3), x_2(1-kx_3)\right] = 0, \quad x_3 \in [0, l]. \tag{1}$$

Here, k is a constant small enough for squares and higher powers to be neglected. The bar B is called almost cylindrical. We assume that B is a bounded regular region (Gurtin (1972), Section 5). We denote by Σ_1 the cross section located at $x_3 = 0$ and by Σ_2 the cross section located at $x_3 = l$. We call ∂B the boundary of B and denote by n_i the components of the outward unit normal of ∂B.

The linear strain measure is defined by

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2}$$

where u_i is the components of the displacement vector. Let t_{ij} be the stress tensor and let h_i be the equilibrated stress vector. The surface force and the equilibrated surface force at a regular point of ∂B are given by

$$t_i = t_{ji}n_j, \; h = h_j n_j, \tag{3}$$

respectively. Throughout this paper, we assume that the body is homogeneous and isotropic. The constitutive equations are given by Cowin and Nunziato (1983)

$$\begin{aligned} t_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\varphi\delta_{ij}, \\ h_j &= \alpha\varphi_j, \quad g = -be_{rr} - \xi\varphi, \end{aligned} \tag{4}$$

where δ_{ij} is the Kronecker delta, φ is the volume distribution function, g is intrinsic equilibrated body force, and λ, μ, b, α, and ξ are constitutive constants.

The equilibrium equations of porous bodies, in the absence of the body loads, are given by

$$t_{ji,j} = 0, \quad h_{j,j} + g = 0. \tag{5}$$

We assume that the lateral surface is free from tractions. Thus, we have the following conditions

$$t_{ki}n_k = 0, \quad h_j n_j = 0 \text{ on } \Pi. \tag{6}$$

Let $\mathbf{R} = (R_1, R_2, R_3)$ and $\mathbf{M} = (M_1, M_2, M_3)$ be prescribed vectors representing the resultant force and resultant moment about O of the tractions acting on Σ_1. On Σ_2, there are tractions applied so as to satisfy the equilibrium conditions of the body. On the end located at $x_3 = 0$, we have the conditions

$$\int_{\Sigma_1} t_{3j}da = -R_j, \quad \int_{\Sigma_1} \varepsilon_{ijk}x_j t_{3k}da = -M_i, \tag{7}$$

where ε_{ijk} is the alternating symbol. We note that there is no contribution of the equilibrated surface force in the resultant force and resultant moment (Ciarletta and Iesan 1993). The elastic potential corresponding to the considered continuum is

$$2W = \lambda e_{rr}e_{jj} + 2\mu e_{ij}e_{ij} + b\varphi + \xi\varphi^2 + \alpha\varphi_{,j}\varphi_{,j}. \tag{8}$$

Throughout this paper, we assume that W is a positive definite quadratic form in the variables e_{ij}, φ, and $\varphi_{,j}$. Then, following Cowin and Nunziato (1983)

$$\mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad 3\lambda + 2\mu > 0, \; (3\lambda + 2\mu)\xi > 3b^2. \tag{9}$$

Methods

In this section, we prove that the problem can be reduced to the study of the deformation of a right cylinder. Let us consider the transformation

$$y_1 = x_1(1-kx_3), \; y_2 = x_2(1-kx_3), \quad y_3 = x_3. \tag{10}$$

By Eq. (10), the domain B is mapped into the right cylinder D, of length l, with the lateral surface S given by

$$F(y_1, y_2) = 0, \quad y_3 \in [0, l]. \tag{11}$$

From Eq. (10), we obtain

$$x_1 = y_1\left(1 + ky_3\right), \quad x_2 = y_2\left(1 + ky_3\right), \; x_3 = y_3. \tag{12}$$

If f is a function of point of class C^1, then

$$\begin{aligned} &\frac{\partial f}{\partial x_1} = \left(1 - ky_3\right)\frac{\partial f}{\partial y_1}, \quad \frac{\partial f}{\partial x_2} = \left(1 - ky_3\right)\frac{\partial f}{\partial y_2}, \\ &\frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial y_3} - k\left(y_1\frac{\partial f}{\partial y_1} + y_2\frac{\partial f}{\partial y_2}\right). \end{aligned} \tag{13}$$

Let us introduce the notation

$$f_{;j} = \frac{\partial f}{\partial y_j}.$$

We seek the solution of the Saint-Venant problem in the form

$$u_i = u_i^{(1)} + ku_i^{(2)}, \quad \varphi = \varphi^{(1)} + k\varphi^{(2)}, \tag{14}$$

where $u_i^{(\alpha)}$ and $\varphi^{(\alpha)}$, $(\alpha = 1, 2)$, are unknown functions. We denote

$$e_{ij}^{(\alpha)} = \frac{1}{2}(u_{i;j}^{(\alpha)} + u_{j;i}^{(\alpha)}). \tag{15}$$

It follows from Eqs. (2), (13), and (15) that

$$\begin{aligned} e_{ij} &= e_{ij}^{(1)} + ke_{ij}^{(2)} + kE_{ij}, \\ \varphi_{,j} &= \varphi_{;j}^{(1)} + k\varphi_{;j}^{(2)} + k\zeta_j, \end{aligned} \tag{16}$$

where

$$E_{11} = -y_3 e^{(1)}_{11},\ E_{22} = -y_3 e^{(1)}_{22},\ E_{33} = -y_\rho u^{(1)}_{3;\rho},$$
$$E_{12} = E_{21} = -y_3 e^{(1)}_{12},\ E_{13} = E_{31} = -\frac{1}{2}\left(y_\rho u^{(1)}_{1;\rho} + y_3 u^{(1)}_{3;1}\right),$$
$$E_{23} = E_{32} = -\frac{1}{2}\left(y_\rho u^{(1)}_{2;\rho} + y_3 u^{(1)}_{3;2}\right),$$
$$\zeta_1 = -y_3 \varphi^{(1)}_{;1},\ \zeta_2 = -y_3 \varphi^{(1)}_{;2},\ \zeta_3 = -y_\rho \varphi_{;\rho}. \tag{17}$$

In view of Eq. (16), from Eq. (4), we obtain

$$t_{ij} = t^{(1)}_{ij} + k t^{(2)}_{ij} + k P_{ij}, \tag{18}$$
$$h_j = h^{(1)}_j + k h^{(2)}_j + k H_j,\ g = g^{(1)} + k g^{(2)} + k G,$$

where we have used the notations

$$t^{(\rho)}_{ij} = \lambda e^{(\rho)}_{rr}\delta_{ij} + 2\mu e^{(\rho)}_{ij} + b\varphi^{(\rho)}\delta_{ij}, \tag{19}$$
$$h^{(\rho)}_i = \alpha \varphi^{(\rho)}_{;i},\ g^{(\rho)} = -b e^{(\rho)}_{rr} - \xi \varphi^{(\rho)},\quad (\rho = 1, 2)$$

and

$$P_{ij} = \lambda E_{rr}\delta_{ij} + 2\mu E_{ij}, \tag{20}$$
$$H_\alpha = -y_3 \varphi^{(1)}_{3;\alpha},\ H_3 = y_\rho \varphi^{(1)}_{;\rho},\ G = -b E_{rr}.$$

The equilibrium equations, to a second order approximation, reduce to the following systems

$$t^{(1)}_{ji;j} = 0,\ h^{(1)}_{j;j} + g^{(1)} = 0, \tag{21}$$

and

$$t^{(2)}_{ji;j} + P_{ji;j} - y_3 t^{(1)}_{\rho i;\rho} - y_\rho t^{(1)}_{3i;\rho} = 0,$$
$$h^{(2)}_{j;j} + g^{(2)} + H_{j;j} - y_3 h^{(1)}_{\alpha;\alpha} + G = 0, \tag{22}$$

on D.

If the direction cosines of the exterior normal to the lateral surface S are $(N_1, N_2, 0)$, then, we have

$$n_1 = N_1,\ \ n_2 = N_2,\ \ n_3 = -k y_\rho N_\rho. \tag{23}$$

It follows from Eqs. (18) and (23) that the boundary conditions (6) are satisfied if we have

$$t^{(1)}_{\alpha i} N_\alpha = 0,\ \ h^{(1)}_\alpha N_\alpha = 0\ \text{ on } S \tag{24}$$

and

$$t^{(2)}_{\alpha i} N_\alpha = -P_{\alpha i} N_\alpha + y_\rho N_\rho t^{(1)}_{i3},$$
$$h^{(2)}_\alpha N_\alpha = -H_\rho N_\rho + h^{(1)}_3 y_\rho N_\rho,\ \text{ on } S. \tag{25}$$

We denote by Σ^*_1 the cross section of D located at $y_3 = 0$. In view of Eqs. (10) and (18), the conditions (7) reduce to

$$\int_{\Sigma^*_1} t^{(1)}_{3i} da = -R_i,\ \int_{\Sigma^*_1} \varepsilon_{ijk} y_j t^{(1)}_{3k} da = -M_i, \tag{26}$$

and

$$\int_{\Sigma^*_1} t^{(2)}_{3i} da = 0,\ \int_{\Sigma^*_2} \varepsilon_{ijk} y_j t^{(2)}_{3k} da = 0. \tag{27}$$

We note that the functions $u^{(1)}_i$ and $\varphi^{(1)}$ satisfy the equations and the boundary conditions in the Saint-Venant problem characterized by Eqs. (15), (19), and (21) on D and the boundary conditions (24) and (26). The functions $u^{(1)}_i$ and $\varphi^{(1)}$ can be determined by using the method given by Ciarletta and Iesan (1993). To find the functions $u^{(2)}_i$ and $\varphi^{(2)}$, we have to solve Eqs. (15), (19), and (22) on D and the boundary conditions (25) and (27). In this problem, the body loads and the surface tractions on the lateral surface S are, in general, different from zero. The functions $u^{(2)}_i$ and $\varphi^{(2)}$ satisfy a problem of Almansi type, in which the body loads and surface forces depend on the functions $u^{(1)}_i$ and $\varphi^{(1)}$. A general method to solve the Almansi problem has been presented by Ieşan and Scalia (2009).

Results and discussion

In this section, we use the above method in order to study the extension of a conical frustum made of a porous elastic material. We consider that the domain B is the interior of a circular cone frustum, bounded by plane ends perpendicular to the axis of the cone (Fig. 1). We choose the rectangular cartesian coordinate frame such that x_3-axis can be the axis of the cone. We assume that the ends Σ_1 and Σ_2 are circles of radius r_1 and r_2, respectively, and that $r_1 < r_2$. The lateral surface Π is defined by

$$x_1^2 + x_2^2 = r_1^2\left(1 + r_1^{-1} x_3 \operatorname{tg}\theta\right)^2,\ \ x_3 \in [0, l], \tag{28}$$

where θ is the angle between the generator and the axis of the cone.

We denote

$$k = \frac{1}{r_1}\operatorname{tg}\theta. \tag{29}$$

We suppose that k is small enough for squares and high powers to be neglected. Then, Eq. (28) can be expressed as

$$x_1^2 (1 - k x_3)^2 + x_2^2 (1 - k x_3)^2 = r_1^2,\quad x_3 \in [0, l]. \tag{30}$$

The surface S is described by

$$y_1^2 + y_2^2 = r_1^2,\quad y_3 \in [0, l]. \tag{31}$$

Thus, the domain D is a right circular cylinder,

$$D = \left\{(y_1, y_2, y_3), y_1^2 + y_2^2 < r_1^2, 0 < y_3 < l\right\}.$$

We assume that $R_\alpha = 0$, $R_3 \neq 0$ and $M_j = 0$. In this case, the conditions (26) reduce to

$$\int_{\Sigma^*_1} t^{(1)}_{3\alpha} da = 0,\ \int_{\Sigma^*_1} x_\alpha t^{(1)}_{33} da = 0,\ \int_{\Sigma^*_1} \varepsilon_{\alpha\beta 3} x_\alpha t^{(1)}_{\beta 3} da = 0, \tag{32}$$

and

$$\int_{\Sigma^*_1} t^{(1)}_{33} da = -R_3. \tag{33}$$

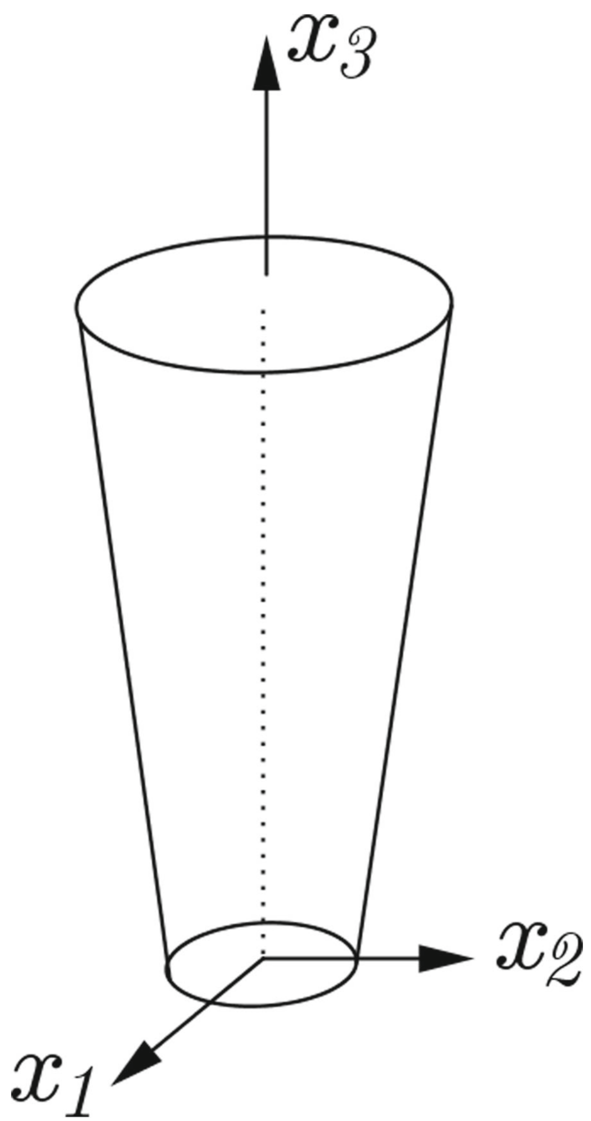

Fig. 1 A conical frustum

Let us determine the functions $u_i^{(\alpha)}$ and $\varphi^{(\alpha)}$, $(\alpha = 1, 2)$. We seek the functions $u_j^{(1)}$ and $\varphi^{(1)}$ in the form

$$u_\alpha^{(1)} = a_3 C_1 y_\alpha,\ u_3^{(1)} = a_3 y_3,\ \varphi^{(1)} = a_3 C_2, \tag{34}$$

where C_α and a_3 are unknown constants. It follows from Eqs. (34), (15), and (19) that

$$\begin{aligned} &t_{11}^{(1)} = t_{22}^{(1)} = [\lambda\,(1 + 2C_1) + 2\mu C_1 + bC_2]\, a_3,\quad t_{12}^{(1)} = 0,\\ &t_{33}^{(1)} = (\lambda + 2\mu + 2\lambda C_1 + bC_2)\, a_3,\quad t_{\alpha 3}^{(1)} = 0,\\ &g^{(1)} = -\,[(1 + 2C_1)\, b + \xi C_2]\, a_3,\quad h_j^{(1)} = 0. \end{aligned} \tag{35}$$

The boundary conditions (24) are satisfied only if

$$2(\lambda + \mu)C_1 + bC_2 = -\lambda. \tag{36}$$

The equilibrium Eq. (21) reduces to

$$2bC_1 + \xi C_2 = -b. \tag{37}$$

It follows from Eqs. (36) and (37) that

$$C_1 = \frac{\lambda\xi - b^2}{2b^2 - 2\xi(\lambda + \mu)},\quad C_2 = \frac{b\mu}{b^2 - \xi(\lambda + \mu)}. \tag{38}$$

We note that the conditions (9) imply that $b^2 - \xi(\lambda + \mu)$ is different from zero. The conditions (32) are identically satisfied. From Eqs. (33) and (35), we find the constant a_3,

$$a_3 = \frac{R_3\left[b^2 - \xi(\lambda + \mu)\right]}{A\left[(\lambda + 2\mu)(\lambda + \mu)\xi - 3\mu b^2 - \lambda^2\xi\right]}, \tag{39}$$

where A is the area of the cross section Σ_1^*.

Thus, the functions $u_i^{(1)}$ and $\varphi^{(1)}$ are determined. Let us study the Almansi problem characterized by Eqs. (15), (19), and (22) on D and the boundary conditions (25) and (27). We note that in the case of extension, we have

$$\begin{aligned} &t_{\alpha\beta}^{(1)} = 0,\ t_{33}^{(1)} = (\lambda + 2\mu + 2\lambda C_1 + bC_2)\, a_3,\\ &h_i^{(1)} = 0,\ g^{(1)} = 0. \end{aligned} \tag{40}$$

From Eqs. (34) and (17), we find that

$$\begin{aligned} &E_{11} = E_{22} = -a_3 C_1 y_3,\ E_{\alpha 3} = -\frac{1}{2} y_\alpha a_3 C_1,\\ &E_{12} = E_{33} = 0,\ \xi_i = 0. \end{aligned} \tag{41}$$

In view of Eqs. (20) and (41), we get

$$\begin{aligned} &P_{11} = P_{22} = -2(\lambda + \mu) a_3 C_1 y_3,\quad P_{12} = 0,\\ &P_{33} = -2\lambda C_1 a_3 y_3,\ P_\alpha = -\mu C_1 a_3 y_\alpha,\\ &H_j = 0,\quad G = 2bC_1 a_3 y_3. \end{aligned} \tag{42}$$

It is a simple matter to see that Eq. (22) reduces to

$$t_{j\alpha;j}^{(2)} = 0,\quad t_{j3;j}^{(2)} - P = 0, \tag{43}$$

$$h_{j;j}^{(2)} + g^{(2)} + G = 0, \tag{44}$$

on D, where

$$P = 2(\lambda + \mu) C_1 a_3. \tag{45}$$

The direction cosines of the exterior normal to the surface (31) are given by $N_\alpha = y_\alpha r_1^{-1}$, $N_3 = 0$. In view of Eqs. (40) and (42), the boundary conditions (25) become

$$\begin{aligned} &t_{\rho\alpha}^{(2)} N_\rho = P r_1^{-1} y_\alpha,\ t_{\rho 3}^{(2)} N_\rho = P_0,\\ &h_\rho^{(2)} N_\rho = 0 \ \text{ for } \ r = r_1, \end{aligned} \tag{46}$$

where

$$P_0 = \left[\lambda + 2\mu + (2\lambda + \mu)\, C_1 + bC_2\right] r_1 a_3. \tag{47}$$

Let us determine the functions $u_i^{(2)}$ and $\varphi^{(2)}$ which satisfy Eqs. (15), (19), (43), and (44) on D, and the boundary conditions (46) and (27). First, we determine the functions u_i^* and φ^* which satisfy Eqs. (15), (19), (43), and (44) on

D and the boundary conditions on the lateral surface (46). We seek these functions in the form

$$u_1^* = A_1 y_1 + B_1 y_1 y_3, \quad u_2^* = A_1 y_2 + B_1 y_2 y_3,$$
$$u_3^* = \frac{1}{2} B_3 y_3^2 + \Psi(y_1, y_2), \ \varphi^* = A_2 + B_2 y_3, \tag{48}$$

on D, where A_α and B_k are unknown constants, and Ψ is an unknown function. The strain tensor e_{ij}^*, corresponding to the displacements u_j^*, is

$$e_{11}^* = e_{22}^* = A_1 + B_1 y_3, \ e_{33} = B_3 y_3,$$
$$e_{12} = 0, \ 2e_{\alpha 3} = \Psi_{;\alpha} + B_1 y_\alpha.$$

The stress tensor t_{ij}^*, the equilibrated stress vector h_j^*, and the intrinsic equilibrated body force g^* associated with the deformation described by Eq. (48) are given by

$$t_{11}^* = t_{22}^* = 2(\lambda + \mu)A_1 + bA_2 + [2(\lambda + \mu)B_1 + bB_2 + \lambda B_3] y_3,$$
$$t_{33}^* = 2\lambda A_1 + bA_2 + [2\lambda B_1 + bB_2 + (\lambda + 2\mu)B_3] y_3, \tag{49}$$
$$t_{12}^* = 0, \ t_{\alpha 3}^* = \mu \left(\Psi_{;\alpha} + B_1 y_\alpha\right),$$
$$h_\alpha^* = 0, \ h_3^* = B_2, \ g^* = -2bA_1 - \xi A_2 - (2bB_1 + \xi B_2 + bB_3) y_3.$$

Equation (44) is satisfied if

$$2bA_1 + \xi A_2 = 0 \tag{50}$$

and

$$2bB_1 + \xi B_2 + bB_3 = 2bC_1 a_3. \tag{51}$$

The first two conditions of Eq. (46) reduce to

$$2(\lambda + \mu)A_1 + bA_2 = P, \tag{52}$$

and

$$2(\lambda + \mu)B_1 + bB_2 + \lambda B_3 = 0. \tag{53}$$

It follows from Eqs. (50) and (52) that

$$A_1 = \frac{P\xi}{2\left[\xi(\lambda + \mu) - b^2\right]}, \quad A_2 = -\frac{bP}{\xi(\lambda + \mu) - b^2}. \tag{54}$$

In view of Eq. (49), the equations of equilibrium (43) reduce to

$$\mu \Delta \Psi = P - 2(\lambda + \mu)B_1 - bB_2 - (\lambda + 2\mu)B_3, \tag{55}$$

on Σ_1^*, where $\Delta U = U_{;\alpha\alpha}$. The third boundary condition of Eq. (46) becomes

$$\mu \Psi_{;\alpha} N_\alpha = P_0 - B_1 \mu y_\rho N_\rho, \quad \text{for } r = r_1. \tag{56}$$

The necessary and sufficient condition to solve the Neumann problem, (55) and (56), is

$$2\lambda B_1 + bB_2 + (\lambda + 2\mu)B_3 = \Lambda, \tag{57}$$

where

$$\Lambda = P - 2r_1^{-1} P_0 = 2a_3 \left(\lambda + 2\mu + \lambda C_1 + bC_2\right). \tag{58}$$

The constants B_k are determined by the system (50), (53), and (57). We obtain

$$B_1 = \frac{1}{\mathcal{D}} \left[(b^2 - \xi\lambda)\Lambda - 4\mu b^2 a_3 C_1\right],$$
$$B_2 = \frac{2}{\mathcal{D}} \left[2ba_3(\lambda + \mu)(\lambda + 2\mu)C_1 - \mu b\Lambda - 2\lambda^2 ba_3 C_1\right], \tag{59}$$
$$B_3 = \frac{2}{\mathcal{D}} \left\{\Lambda \left[\xi(\lambda + \mu) - b^2\right] - 2\mu b^2 a_3 C_1\right\},$$

where

$$\mathcal{D} = 2\mu \left[(3\lambda + 2\mu)\xi - 3b^2\right]. \tag{60}$$

We note that the relations (9) imply that $\mathcal{D} \neq 0$. The last condition from Eq. (46) is identically satisfied.

The solution of the boundary value problem (55) and (56) is given by

$$\Psi = \frac{1}{4\mu} \left[P - 2(\lambda + \mu)B_1 - bB_2 - (\lambda + 2\mu)B_3\right] r^2, \tag{61}$$

where $r^2 = y_\alpha y_\alpha$.

We seek the functions $u_i^{(2)}$ and $\varphi^{(2)}$ in the form

$$u_i^{(2)} = u_i^* + v_i, \quad \varphi^{(2)} = \varphi^* + \psi. \tag{62}$$

It is a simple matter to see that the functions v_i and ψ must satisfy the equilibrium equations in the absence of the body loads and the conditions on the lateral surface S, in the absence of superficial forces. Let s_{ij} be the stress tensor corresponding to the displacements v_i and volume distribution function ψ. It follows from Eqs. (49), (61), and (62) that the conditions (27) reduce to

$$\int_{\Sigma^*} s_{3\alpha} da = 0, \quad \int_{\Sigma_1^*} s_{33} da = -R_3^*, \quad \int_{\Sigma_1^*} \varepsilon_{ijk} y_j s_{3k} da = 0 \tag{63}$$

where

$$R_3^* = (2\lambda A_1 + bA_2)A. \tag{64}$$

We conclude that v_i and ψ satisfy a problem of extension with the axial force R_3^*. These functions are given by

$$v_\alpha = d_3 C_1 y_\alpha, \ v_3 = d_3 y_3, \ \psi = d_3 C_2, \tag{65}$$

where C_α is defined by Eq. (38) and

$$d_3 = \frac{R_3^* \left[b^2 - \xi(\lambda + \mu)\right]}{A\left[(\lambda + 2\mu)(\lambda + \mu)\xi - 3\mu b^2 - \lambda^2 \xi\right]}. \tag{66}$$

It follows from Eqs. (10), (14), (34), (62), and (65) that the solution of the problem of extension of the porous conical frustum is given by

$$u_\alpha = x_\alpha \left[a_3 C_1 + k\left(A_1 + d_3 C_1\right) + k(B_1 - a_3 C_1)x_3\right],$$
$$u_3 = \left(a_3 + kd_3\right) x_3 + \frac{1}{2} kB_3 x_3^2 + kQ x_\alpha x_\alpha, \tag{67}$$
$$\varphi = \left(a_3 + kd_3\right) C_2 + k\left(A_2 + B_2 x_3\right), \quad (x_1, x_2, x_3) \in B,$$

where we have used the notation

$$Q = \frac{1}{2\mu}\left[(\lambda + \mu)C_1 a_3 - \mu B_3\right].$$

The cartesian coordinate frame consists of the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and the origin O. Let $R_3 = -F$, where F is a positive constant. In this case, the resultant force of the tractions acting on the end located at $x_3 = l$ is $F\mathbf{e}_3$ and the point O is fixed. Let x_i be the coordinates of the point X_0 in the reference configuration, and let ξ_i be the coordinates of the corresponding point X in the deformed configuration. Then, we have $\xi_i = x_i + u_i$, and from Eq. (67), we obtain

$$\begin{aligned} \xi_\alpha &= (1 + T_1 + T_2 x_3)\, x_\alpha, \\ \xi_3 &= (1 + \Omega_1)\, x_3 + \Omega_2 x_3^2 + kQ x_\alpha x_\alpha, \end{aligned} \tag{68}$$

where

$$\begin{aligned} &T_1 = a_3 C_1 + k\,(d_3 C_1 + A_1)\,, \quad T_2 = k\,(B_1 - a_3 C_1)\,, \\ &\Omega_1 = a_3 + k d_3, \quad \Omega_2 = \frac{1}{2} k B_3. \end{aligned} \tag{69}$$

The point Y_0 which, prior to deformation, had the coordinates $(0, 0, l)$ goes into point Y from the deformed configuration. From Eq. (68), we find that the point Y has the coordinates $(0, 0, l^*)$ where

$$l^* = (1 + \Omega_1 + \Omega_2 l)\, l. \tag{70}$$

We note that in the classical elasticity, the value of l^* for a homogeneous right cylinder is $(1 + F/E\mathcal{A})l$, where E is Young's modulus and $\mathcal{A}$ is the area of the cross section. Let us choose magnesium crystal as the hypothetical material for which the values of the constitutive coefficients are (Bachher, 2015)

$$\begin{aligned} &\lambda = 2.17 \times 10^{10} N \cdot m^{-1}, \mu = 3.278 \times 10^{10} N \cdot m^{-1}, \\ &\alpha = 3.688 \times 10^{-5} N, \quad b = 1.13849 \times 10^{10} N \cdot m^2, \\ &\xi = 1.475 \times 10^{10} N \cdot m^{-2}. \end{aligned}$$

If $\mathbf{R} = -\mathbf{e}_3$ and $k = 5.10^{-1}$, then from Eq. (70), we get

$$l^* = (1.01063 + 0.00011 l) 10^{-6} l.$$

Let us consider a circle $(\mathcal{C})$ of radius ρ, located at the plane $x_3 = \eta$, where ρ and η are given constants. It follows from Eq. (68) that the image of $(\mathcal{C})$ in the deformed configuration is the circle

$$\xi_1^2 + \xi_2^2 = \rho^2\,(1 + T_1 + T_2 c)^2\,, \tag{71}$$

located at the plane $\xi_3 = (1 + \Omega_1)\,\eta + \Omega_2 \eta^2 + kQ\rho^2$. The relation (71) can be used to describe the deformation of the surface S. Let us assume that r_1 =10 mm. We consider the circle of radius r_1 located at the plane $x_3 = c$. Let (Γ) be the image of this circle in the deformed configuration. We denote by $R(c)$ the radius of (Γ). The variation of $R(c)$ with respect to variable c is presented in Fig. 2. The material parameters used are the same as in the previous example.

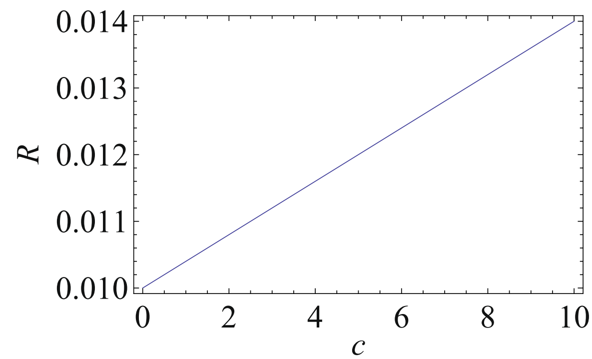

Fig. 2 Variation of $R(c)$

We note that in the case of the problem of extension of a right cylinder, the displacements and the volume fraction field depend on the coordinates x_i at most linearly.

Conclusions

The results established in this paper can be summarized as follows:

a) We have studied the Saint-Venant problem for an almost cylindrical bar made of a porous elastic material. The problem was reduced to the solving of a problem of Saint-Venant type for a right cylinder D and to the problem of Almansi for D.
b) We have used the above method to investigate the behavior of a conical frustum subjected to extension. In this case, the problem reduces to a problem of Almansi type for a right circular cylindrical. The displacement vector field and the volume fraction field have been determined. The displacement vector is a polynomial of two degree in the cartesian coordinate. The volume fraction field depends linearly on the axial coordinate.
c) The salient feature of the solution of the problem of extension is that the displacement vector field and the stresses contain new terms characterizing the influence of the material porosity, and their values are therefore modified from the values predicted by the classical elasticity.

Competing interests
The author declares that he has no competing interests.

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