Lemoubou and Kamdem *International Journal of Mechanical and Materials Engineering* (2017) 12:17
DOI 10.1186/s40712-017-0084-4

International Journal of Mechanical and Materials Engineering

ORIGINAL PAPER Open Access

A quasi-analytical solution of homogeneous extended surfaces heat diffusion equation

Ernest Léontin Lemoubou and Hervé Thierry Tagne Kamdem*

Abstract

Background: In this study, a quasi-analytical solution for longitudinal fin and pin heat conduction problems is investigated.

Methods: The differential transform method, which is based on the Taylor series expansion, is adapted for the development of the solution. The proposed differential transform solution uses a set of mathematical operations to transform the heat conduction equation together with the fin profile in order to yield a closeform series of homogeneous extended surface heat diffusion equation.

Results and conclusions: The application of the proposed differential transform method solution to longitudinal fins of rectangular and triangular profiles and pins of cylindrical and conical profiles heat conduction problems showed an excellent agreement on both fin temperature and efficiencies when compared to exact results. Therefore, the proposed differential transform method can be useful for optimal design of practical extended surfaces with suitable profile for temperature response.

Keywords: Analytical solution, Differential transform method, Heat conduction, Longitudinal fin, Pins

Background

Extended surfaces in the forms of longitudinal or radial fins or spines with various cross sections are widely used in industrial applications such as air-cooled craft engines, cooling of computer processors and electrical components, air-conditioning, refrigeration, heat exchangers, and solar collectors. These devices, which provide a considerable increase in the surface area for heat transfer between a heated source and a cooler ambient fluid, are most effective to enhance heat transfer between a surface and an adjacent fluid (Kraus et al. 2001). In designing extended surfaces, the first step consists of assuming that the heat transfer is governed by a one-dimensional homogeneous steady conduction along an extended surface and a uniform convection at the surface area (Arauzo et al. 2005; Kang 2009; Brestovic et al. 2015). Although an exact solution for this problem can be found in the literature, it may be either difficult or not possible to obtain when designing practical extended surfaces with profile matching suitable geometry for temperature response. Moreover, the exact solution of some problems requires difficult manipulation of special functions such as Bessel functions.

Several efficient analytical or quasi-analytical methods have been developed over the past two decades to solve linear and nonlinear problems in science and engineering including the power series, the Adomian decomposition, the homotopy, and the differential transformation methods (Diez et al. 2009; Aziz and Bouaziz 2011; Torabi and Zhang 2013; Hayat et al. 2016). In general, exact solutions are mostly based on the tedious computation of special functions (Kraus et al. 2001; Diez et al. 2009). The power series method has been used by Arauzo et al. (2005) to approximate the solution of the one-dimensional steady heat conduction equation that governs the temperature variation in annular fins of hyperbolic profile. This method is a standard technique for solving linear ordinary differential equations with variable coefficients. The Adomian decomposition method has been used by Arslanturk (2005) and Bhowmik et al. (2013) to compute a closed-form solution for a straight convecting rectangular and hyperbolic profile annular fin with temperature-dependent thermal conductivity. This method represents the solution by an infinite series of the so-called Adomian polynomials and uses an iterative method such as the Newton–Raphson for the evaluation of the undetermined temperature at the fin tip.

* Correspondence: herve.kamdem@univ-dschang.org
Laboratory of Mechanics & Modeling of Physics Systems, Department of Physics/Faculty of Science, University of Dschang, P.O. Box 67, Dschang, Cameroon

The homotopy method is a quasi-analytical solution for solving nonlinear boundary value problem (Domairry and Fazeli 2009; Hayat et al. 2017a). This method does not require the calculation of Adomian polynomials as required for the Adomian decomposition method, but it requires an initial approximation (Roy and Mallick 2016; Hayat et al. 2017b; Waqas et al. 2016). Inc (2008) used the homotopy analysis method to evaluate the efficiency of straight fins with temperature-dependent thermal conductivity and determined the temperature distribution within the fin. The results show that the homotopy analysis method presents faster convergence and higher accuracy than the Adomian decomposition method and the homotopy perturbation method for nonlinear problems in science and engineering. The differential transform method (DTM) is based on the Taylor series expansion and constructs an analytical solution in the form of a polynomial. In general, the solution necessitates an iterative method such as the Newton–Raphson methods for the determination of the initial temperature transform function value. Joneidi et al. (2009) applied the differential transform method to predict temperature and efficiency of convective straight fins with temperature-dependent thermal conductivity. Their obtained results compared to exact and numerical results reveal that the differential transform method is an effective and accurate method for analyzing extended surfaces' nonlinear heat transfer problems. Kundu and Lee (2012) determined the performance of different fin geometries by analyzing heat transfer in rectangular, triangular, convex, and exponential geometric longitudinal fins using the differential transform method. These authors demonstrated that the differential transform is precise and cost efficient for analyzing nonlinear heat and mass transfer effects in extended surfaces.

The literature presents several quasi-analytical methods for analyzing heat transfer problems in extended surfaces. The different quasi-analytical methods, which used an iterative method such as the Newton–Raphson for the evaluation of an undetermined parameter, are applied mostly for nonlinear problems. On the other hand, the solution of linear problems is generally obtained as a particular case of nonlinear problems. The contribution of the present work is to present close-form series solution of the homogeneous extended surface heat diffusion equation using the differential transform method. This can be a useful strategy in developing an analytical solution when designing practical extended surfaces with suitable geometry for temperature response. The proposed differential transform solution uses a set of mathematical operations to transform the heat conduction equation together with the fin profile in order to yield a close-form series of homogeneous extended surface heat diffusion equation which avoid using an iterative method. The homogeneous extended surfaces in the forms of longitudinal fins of rectangular and triangular profiles and pins of cylindrical and conical profiles are attached to a primary surface at constant temperature heat losses by convection to the surrounding medium and where the heat loss from the tip of the extended surfaces is assumed to be negligible. The temperature distribution and efficiency within extended surfaces are analyzed and compared against exact results.

Methods

Problem description

The problem under consideration consists of four homogeneous extended surfaces, longitudinal fins of rectangular and triangular profiles and pins of cylindrical and conical profiles as shown in Fig. 1. These extended surfaces are attached to a primary surface at constant temperature heat losses by convection to the surrounding medium and where the heat loss from the tip is assumed to be negligible.

For most engineering problems, the extended surface temperature must be maintained lower than the surrounding air to be cooled. Therefore, the extended surface which is attached to a primary surface at constant temperature T_b loses heat by convection to the surrounding medium of temperature T_∞. In this work, the heat loss from the tip of the extended surface is assumed to be negligible and the heat conduction is assumed to occur solely in the longitudinal direction. The governing of the conduction equations for this problem is well described in the literature (Kraus et al. 2001; Yaghoobi and Torabi 2011; Kundu and Lee 2012). For the problem under consideration, the governing equation for $0 \le x \le b$ and boundary conditions can be written as

$$\frac{d}{dx}\left(k_c\, p^{\xi} \frac{dT}{dx}\right) - \xi\, p^{(\xi-1)}\, h[T(x) - T_\infty] = 0 \quad (1)$$

$$T\rfloor_{x=b} = T_b, \qquad A(x)\frac{dT}{dx}\rfloor_{x=0} = 0 \quad (2)$$

where k_c is the constant thermal conductivity, h is the constant convective coefficient of the cooled air, A is the fin section area, the parameter $\xi = 1$ for longitudinal fins and $\xi = 2$ for pins and the extended surfaces profile, $p(x)$, is given by (Kraus et al. 2001)

$$p(x) = \begin{cases} \dfrac{w}{2}\left(\dfrac{x}{b}\right)^{(1-2\gamma)(1-\gamma)} & , if\ longitudinal \\ \dfrac{w}{2}\left(\dfrac{x}{b}\right)^{(1-2\gamma)/(2-\gamma)} & , if\ pins \end{cases} \quad (3)$$

with $\gamma = 1/2$ for fins of rectangular and cylindrical profile and $\gamma = 0$ for triangular profile and $\gamma = -1$ for pins or

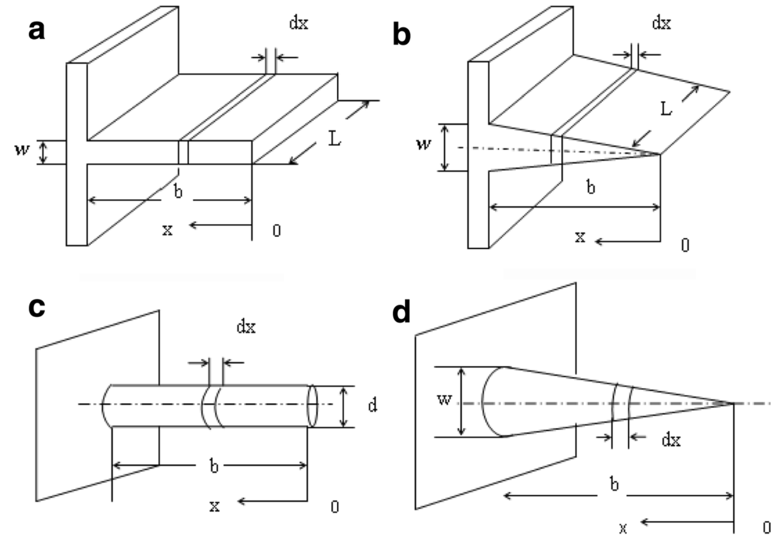

Fig. 1 Sketch diagram of longitudinal fins of rectangular (**a**) and triangular profile (**b**) and pins of cylindrical (**c**) and conical profile (**d**)

conical profile. Let us consider the following dimensionless temperature and coordinate, respectively, as

$$\theta = \frac{T(x)-T_\infty}{T_b-T_\infty}, \qquad \zeta = \frac{x}{b}, \tag{4}$$

The governing Eqs. (1) for and (2) can be written in dimensionless form as

$$\frac{d}{d\zeta}\left[p^\xi \frac{d\theta}{d\zeta}\right] - \frac{\xi h b^2}{k_c} p^{(\xi-1)}\theta = 0, \quad 0 \le \zeta \le 1 \tag{5}$$

$$\theta(\zeta=1)=1, \quad A(\zeta)\ \frac{d\theta}{d\zeta}(\zeta=0)=0 \tag{6}$$

Equation (6) can be further written as

$$p\frac{d^2\theta}{d\zeta^2} + \xi\frac{dp^\xi}{d\zeta}\frac{d\theta}{d\zeta} - \frac{\xi h b^2}{k_c}\theta = 0 \tag{7}$$

Differential transform operations

The differential transform method is generally used to derive a solution for a wide class of linear and nonlinear ordinary differential equations in the form of Taylor series (Chen and Ho, 1999; Hassan 2002). Consider an analytical function $f(x)$ defined in the domain D. The development of the function $f(x)$ around a point $x = x_i$ in Taylor series can be represented as (Jang et al. 2010; Chen and Ju 2004)

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-x_i)^k}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_i} \tag{8}$$

It should be noted that for the development around point $x = 0$, Eq. (9) leads to Maclaurin series. The function $F(k)$ which represents the transform form of the original function $f(x)$ can be expressed by the following relation (Jang et al. 2010; Chen and Ju 2004)

$$F(k) = \frac{(H)^k}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_i}, k = 0, 1, \cdots, \infty \tag{9}$$

where H is the space horizon of interest. Considering that the space horizon is $H = 1$, the transform function $F(k)$ is related to $f(x)$ by the differential inverse transform (Hassan 2002; Chen and Ju 2004)

$$f(x) = \quad \sum_{k=0}^{\infty} (x-x_i)^k F(k) \tag{10}$$

If we consider that $f(x)$ is developed in the Taylor series around point $x = 0$, the different operators between $f(x)$ and the transform function $F(k)$ of Table 1 can be derived (Hassan 2002; Joneidi et al. 2009;

Table 1 One-dimensional DTM fundamental operations

Functions	Transform functions
$f(x)=$	$F(k)$
$af_1(x) + bf_2(x)$	$F(k) = aF_1(k) + bF_2(k)$
$\frac{df}{dx}$	$F(k) = (k+1)F(k+1)$
$\frac{d^2f}{dx^2}$	$F(k) = (k+1)(k+2)F(k+2)$
x^n	$F(k) = \delta(k-n) = \begin{cases} 1 & si \quad k = n \\ 0 & si \quad k \neq n \end{cases}$
$\exp(\lambda x)$	$F(k) = \frac{\lambda^k}{k!}$
$f_1(x)f_2(x)$	$F(k) = \sum_{l=0}^{k} F_1(l)F_2(k-l)$
$f_1(x)f_2(x)f_3(x)$	$F(k) = \sum_{h=0}^{k}\ \sum_{l=0}^{h} F_1(l)F_2(h-l)F_3(k-h)$

Yaghoobi and Torabi 2011). The key feature of the DTM is applied by using the operator of Table 1 to the ordinary differential equation of heat conduction through the extended surfaces. An easy way to derive the differential transform of Eqs. (6) and (7) is to find the differential transform of each term.

Considering Table 1 operators, the differential transform of Eq. (7) for longitudinal fins and pins gives, respectively

$$(k+1)(k+2)P(k)\Theta(k+2)+\sum_{l=0}^{k}P(l)\Theta(l)(k-l+2)\times(k-l+1)\Theta(k-l+2)-\frac{h}{k_c}\Theta(k)\ \ =0 \quad (11)$$

$$2\sum_{l=0}^{k}(l+1)(k-l+1)P(l+1)\Theta(k-l+1)+\sum_{h=0}^{k}\sum_{l=0}^{h}(k-h+1)(k-h+2)P(l)P(h-l)\ \Theta(k-h+2)-\frac{2hb^2}{k_c}\Theta(k)=0 \quad (12)$$

Taking the differential transform of the boundary conditions, Eq. (5), it can be obtained, respectively

$$\Theta(1)=0 \quad (13)$$

$$\sum_{k=0}^{\infty}\Theta(k)=1, \quad (14)$$

The transform condition (13) is applied only for rectangular and cylindrical fins whereas (14) is adapted for all the geometries.

Results and discussion

For stationary heat conduction through fins and pins with a dry surface and constant thermal conductivity cooled by air of constant heat transfer coefficient, exact analytical solution using a standard method of ordinary differential equations can be found in the literature (Kraus, et al., 2001; Kundu and Lee 2012). In the present work, the DTM is considered as alternative for obtaining analytical solution and the predictions are compared to the results from the standard method of ordinary differential equations.

Longitudinal fin with rectangular profile

For the first problem, the DTM is applied to a longitudinal fin of rectangular profile cooled by convection to the surrounding medium and with negligible heat loss from the tip. The terminology of this fin is described in Fig. 1a, and the profile function is constant. The transform function of the profile is then

$$P(l)=\frac{w}{2}\delta(l) \quad (15)$$

The temperature transform function is reduced to

$$\Theta(k+2)=\frac{m^2\Theta(k)}{(k+1)(k+2)} \quad (16)$$

with $m=(2h/k_c\,w)^{\frac{1}{2}}$. Considering Eqs. (13) and (16), the temperature transform function can be written in a generalized form as

$$\Theta(2k)=\frac{m^{2k}}{(2k)!}\Theta(0),\quad k=1,2,3,.... \quad (17)$$

$$\Theta(2k+1)=0,\qquad k=0,1,2,3,.... \quad (18)$$

The discrete value $\Theta(0)$ is obtained by applying the second boundary condition Eq. (14) as

$$\Theta(0)=1\Big/\sum_{k=0}^{n}\frac{m^{2k}}{(2k)!} \quad (19)$$

Considering Eq. (10), the temperature distribution for the fin of rectangular profile can then be expressed as

$$\theta(\zeta)=\ \sum_{k=0}^{n}\frac{(m\zeta)^{2k}}{(2k)!}\Big/\sum_{k=0}^{n}\frac{m^{2k}}{(2k)!} \quad (20)$$

The temperature distribution given by Eq. (20) is computed and compared to the exact solution obtained using the standard method of ordinary differential equations (Kraus et al. 2001). The results were truncated to four decimal places since this precision is generally sufficient for engineering problems. The relative errors between the DTM and the exact results for the temperature are presented in Table 2. As expected, the DTM converges towards the exact solution as the order of the approximations is increasing. For both low and high value of the thermal length characteristic parameter, the DTM convergence in four significant figures at relatively low approximation orders: the first fourth terms of the Taylor series is sufficient.

The dimensionless temperature within the longitudinal fin of rectangular profile versus the dimensionless length for the different thermal length characteristic parameter $mb=(2h/k_c\,w)^{\frac{1}{2}}\,b$ can be seen in Fig. 2. Excellent agreement can be observed between DTM and the exact results. It is also noticed that the temperature increases with the height of the rectangular fin for a fixed value of the thermal length and tends to the unit when this fin characteristic is equal to one. For a dimensionless length, the magnitude of temperature

Table 2 Comparison of the DTM and exact results for different approximations order

	$mb = 0.6$			$mb = 1.4$				
	Exact	DTM		Exact	DTM			
ζ		$n = 1$	2		$n = 1$	2	3	4
0.0	0.8436	0.0039	0	0.4649	0.0401	0.0024	0.0001	0
0.1	0.8451	0.0039	0	0.4695	0.0405	0.0024	0.0001	0
0.2	0.8496	0.0039	0	0.4833	0.0416	0.0024	0.0001	0
0.3	0.8573	0.0039	0	0.5065	0.0431	0.0026	0.0001	0
0.4	0.8680	0.0039	0	0.5397	0.0445	0.0027	0.0001	0
0.5	0.8818	0.0038	0	0.5836	0.0452	0.0029	0.0001	0
0.6	0.8988	0.0036	0	0.6388	0.0444	0.0030	0.0001	0
0.7	0.9191	0.0032	0	0.7066	0.0409	0.0030	0.0001	0
0.8	0.9426	0.0025	0	0.7883	0.0335	0.0027	0.0001	0
0.9	0.9696	0.0015	0	0.8855	0.0205	0.0018	0.0001	0
1.0	1.0000	0.0000	0	1.0000	0.0000	0.0000	0.0000	0

decreases with increasing thermal length characteristic parameter. It should be noted that the thermal length characteristic parameter increases with the convection coefficient and decreases with conduction coefficient and fin thickness. In such case, the use of the longitudinal fin of rectangular profile will be optimal for a low convection coefficient of the cooling fluid or if the fin is designed with high thermal conductivity and thickness.

From the Fourier law, the heat transfer rate dissipated from the longitudinal fin to a neighboring fluid can be expressed in a dimensionless variable as

$$q(\zeta) = k_c A \frac{d\theta(\zeta)}{d\zeta} \tag{21}$$

with the fin cross-sectional area $A = wL$ where δ and L are the fin thickness and length, respectively.

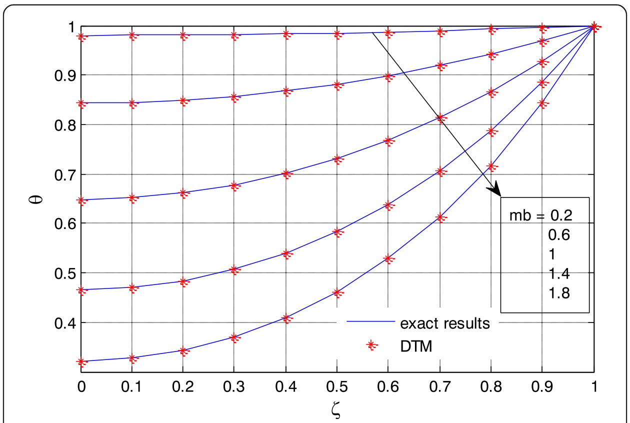

Fig. 2 Temperature evolution for a rectangular fin

Considering the derivative of Eq. (20), the heat transfer rate from Eq. (21) becomes

$$q(\zeta) = k_c wLm^2(T_b - T_\infty) \sum_{k=1}^{n} \frac{m^{2k}\zeta^{2k-1}}{(2k-1)!} \Big/ \sum_{k=0}^{n} \frac{m^{2k}}{(2k)!} \tag{22}$$

The fin efficiency can be evaluated as the ratio of heat transfer rate at the base of the fin to its ideal transfer rate if the entire fin were at the same temperature as its base

$$\eta = \frac{q(\zeta = 1)}{q_{ideal}} = \sum_{k=1}^{n} \frac{m^{2k}}{(2k-1)!} \Big/ \sum_{k=0}^{n} \frac{m^{2k}}{(2k)!} \tag{23}$$

Figure 3 presents the behavior of a longitudinal fin of rectangular profile efficiency against thermal length characteristic parameter. Excellent agreement between the DTM efficiency prediction and exact results is observed. From Fig. 3, it is noted the decrease of efficiency is in accordance with increasing thermal length. This means that the performance of the rectangular fin decreases when the fin has high thickness and conduction coefficient or low convection coefficient of the cooling fluid.

Longitudinal fin of triangular profile

The second test problem considers longitudinal fins of triangular profile cooled by convection to the surrounding medium and with negligible heat loss from its tip. For this fin, shown in Fig. 1b, $\gamma = 0$ and the profile transform function gives

$$P(l) = \frac{w}{2}\delta(l-1) \tag{24}$$

The temperature transform function for the longitudinal fin given by Eq. (11) reduces to

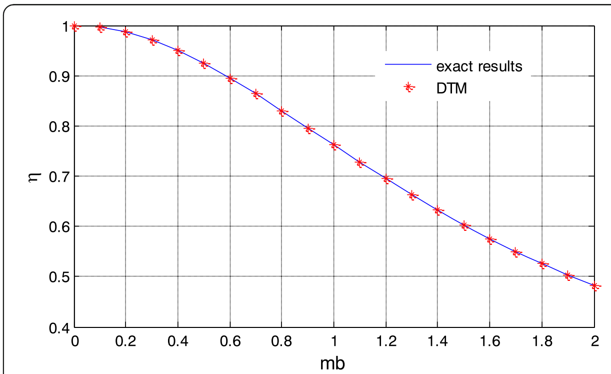

Fig. 3 Efficiency for a rectangular fin

$$\Theta(k+1) = \frac{1}{k+1}\sum_{l=0}^{k}\delta(l-1)(k-l+1)(k-l+2)\Theta(k-l+2) + \frac{m^2}{k+1}\Theta(k) \tag{25}$$

with $m = \sqrt{2\,h/k_c\,w}$. The temperature transform function can be generalized as

$$\Theta(k) = \Theta(0)m^{2k}/(k!)^2, \quad k = 1, 2, 3, \ldots. \tag{26}$$

From the boundary condition, Eq. (14), the value of the constant Θ(0) is given by

$$\Theta(0) = 1\Big/\sum_{k=0}^{n}\frac{m^{2k}}{(k!)^2} \tag{27}$$

Therefore, the temperature distribution for a longitudinal fin of triangular profile using Eq. (10) can be expressed as

$$\theta(\zeta) = \sum_{k=0}^{n}\frac{m^{2k}\zeta^k}{(k!)^2}\Big/\sum_{k=0}^{n}\frac{m^{2k}}{(k!)^2} \tag{28}$$

The temperature computed using Eq. (28) is compared to the exact solutions from the standard method of ordinary differential equations. The DTM solutions converge to the exact solutions for the five terms of the Taylor series, and for all the significant figures of the exact solution. On Fig. 4, it can be seen that the evolution of the dimensionless temperature of the longitudinal fin of triangular profile with the dimensionless length increased as thermal length characteristic parameter increased. The decrease of the fin temperature with the thermal length characteristic parameter indicates that the loss of heat fin is more significant for longitudinal fins of triangular profiles with high thickness and conduction coefficient or for low convection coefficient of the cooling fluid.

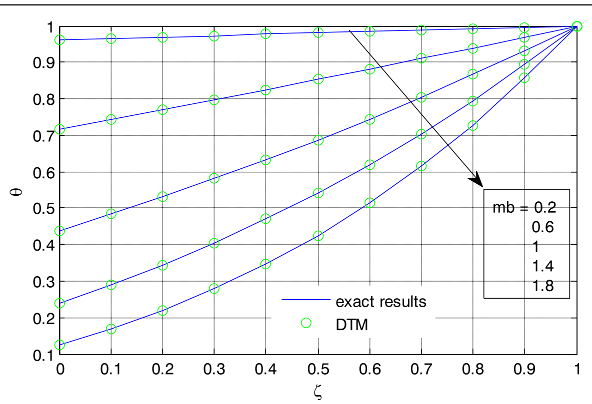

Fig. 4 Temperature evolution for a triangular fin

Considering the heat transfer rate dissipated from the fin to a neighboring fluid, Eq. (21), the heat transfer rate dissipated from the longitudinal fin of triangular profile having cross-sectional area $A = wL$ is

$$\mathrm{q}(\zeta) = k_c\mathrm{wLm}^2(T_b - T_\infty)\sum_{k=1}^{n}\frac{m^{2k}\zeta^{k-1}}{(k-1)!k!}\Big/\sum_{k=0}^{n}\frac{m^{2k}}{(k!)^2} \tag{29}$$

Consequently, the fin efficiency can be expressed as

$$\eta = \frac{q(\zeta=1)}{2hL(T_b - T_\infty)} = \sum_{k=1}^{n}\frac{m^{2k}}{(k-1)!k!}\Big/\sum_{k=0}^{n}\frac{m^{2k}}{(k!)^2} \tag{30}$$

The decrease of the longitudinal fin of triangular profile efficiency with increased thermal length characteristic parameter can be seen in Fig. 5. An excellent agreement between present DTM and exact results is achieved.

Pin of cylindrical profile

A pin of cylindrical profile with the heat loss from the tip which is assumed to be negligible and cooled by convection to the surrounding medium is now considered. Since for this pin, the profile function has the same expression than that for longitudinal fin of rectangular profile, the profile transform function is given by Eq. (15). It can also be deduced from Eq. (12) that the temperature transform function is identical to Eq. (15) where the parameter m is now given by $m = (4h/k_c\,d)^{\frac{1}{2}}$. Therefore, the temperature distribution through the pin of cylindrical profile is given by Eq. (20). The fin cross-sectional area of this pin is $A = \pi d^2/4$ and therefore the heat transfer rate dissipated from the pin of cylindrical profile to the surrounding fluid is

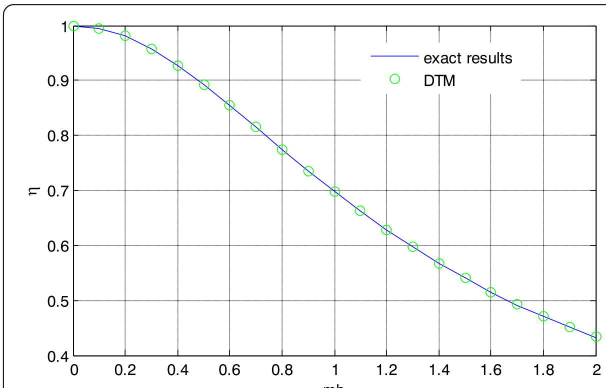

Fig. 5 Efficiency for a triangular fin

$$q(\zeta) = \pi d^2\, k_c m^2 (T_b - T_\infty) \sum_{k=1}^{n} \frac{m^{2k}\zeta^{2k-1}}{(2k-1)!} \Big/ 4 \sum_{k=0}^{n} \frac{m^{2k}}{(2k)!} \tag{31}$$

The efficiency of this fin can be evaluated as

$$\eta = \frac{q(\zeta=1)}{\pi dh(T_b - T_\infty)} = \sum_{k=1}^{n} \frac{m^{2k}}{(2k-1)!} \Big/ \sum_{k=0}^{n} \frac{m^{2k}}{(2k)!} \tag{32}$$

Figures 6 and 7 compare the pins' dimensionless temperature efficiency predicted using DTM and exact results. An excellent agreement between the two methods is observed for DTM with four terms of the Taylor series. It can be seen in Fig. 6 that the dimensionless temperature increases with increasing pin dimensionless length, while the pin efficiency decreased with increasing thermal length characteristic parameter. This indicates that a pin of cylindrical profile dissipates more heat transfer for high values of the pin diameter and thermal conductivity or for low convection coefficient of the cooling fluid.

Pin of conical profile

The last problem deals with a pin of conical profile cooled by convection to the surrounding medium cooled and where the heat loss from the tip is assumed to be negligible. For this pin, shown in Fig. 1d, the profile transform function is expressed as

$$P(l) = \frac{w}{2}\delta(l-1) \tag{33}$$

For the general discrete expression for pin, Eq. (12) is reduced to

$$\Theta(k+1) = \frac{1}{2(k+1)} \left\{ \sum_{l=0}^{k} \delta(l-1)(k-l+1)(k-l+2)\, \Theta(k-l+2) - 2m^2\Theta(k) \right\} \tag{34}$$

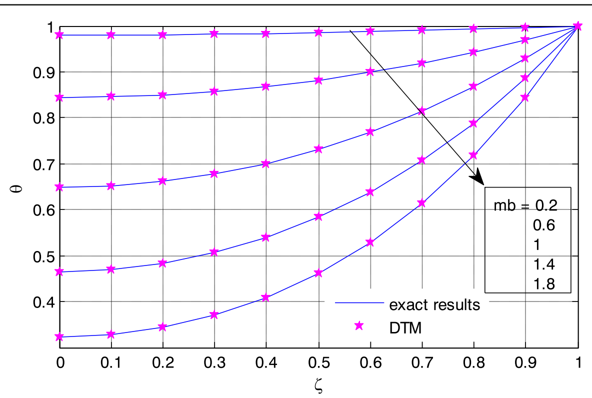

Fig. 6 Temperature evolution for a cylindrical fin

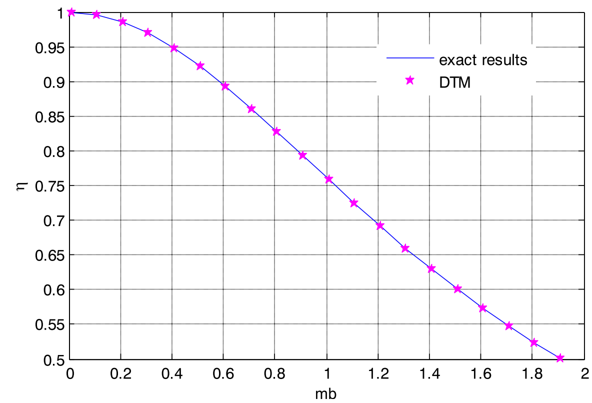

Fig. 7 Efficiency for a cylindrical fin

with $m = (4h/k_c\, w)^{\frac{1}{2}}$. The temperature transform function can be generalized as

$$\Theta(k) = \frac{2^k\, m^{2k}}{k!(k+1)!}\, \Theta(0), \quad k = 1, 2... \tag{35}$$

Application of boundary transform Eq. (14) gives

$$\Theta(0) = \quad 1 \Big/ \sum_{k=0}^{n} \frac{2^k\, m^{2k}}{k!(k+1)!} \tag{36}$$

The temperature distribution within the pin is obtained by substituting Eqs. (35) and (36) in Eq. (10) as

$$\theta(\zeta) = \sum_{k=0}^{n} \frac{2^k\, m^{2k}}{k!(k+1)!}\zeta^k \Big/ \sum_{k=0}^{n} \frac{2^k\, m^{2k}}{k!(k+1)!} \tag{37}$$

Figure 8 shows the variation of the dimensionless temperature according to the pin of the conical profile dimensionless length. Excellent agreement between present DTM using five terms of the Taylor series terms and exact results is observed. It can be seen that the temperature increases with increasing pin length, while

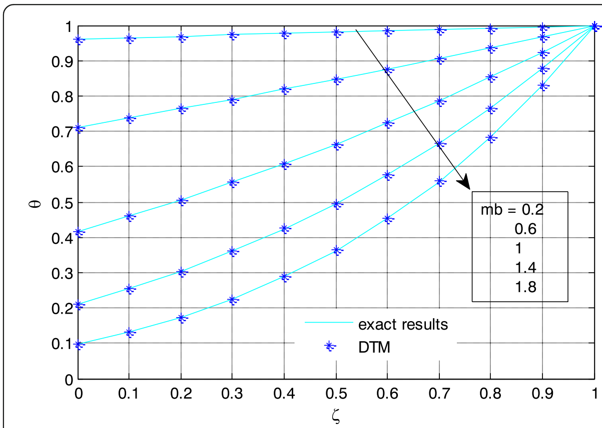

Fig. 8 Temperature evolution for a conical pin

it decreases with increasing thermal length characteristic parameter.

The heat transfer dissipated by the pin having cross-sectional area $A = \pi w^2/4$ can be evaluated as

$$q_b = k_c \frac{\pi w^2}{4} m^2 (T_b - T_\infty) \sum_{k=1}^{n} \frac{2^k m^{2k-2}}{(k-1)!(k+1)!} \Big/ \sum_{k=0}^{n} \frac{2^k m^{2k}}{k!(k+1)!} \tag{38}$$

The pin of conical profile efficiency is given by

$$\eta = \frac{q(\zeta = 1)}{\pi w h (T_b - T_\infty)/2} = \sum_{k=1}^{n} \frac{2^{k+1} m^{2k-2}}{(k-1)!(k+1)!} \Big/ \sum_{k=0}^{n} \frac{2^k m^{2k}}{k!(k+1)!} \tag{39}$$

The decrease of pin efficiency with thermal length characteristic parameter is illustrated in Fig. 9, where an excellent agreement between present DTM and exact results is observed. The variation of temperature and efficiencies with the thermal length characteristic parameter confirms that the performance of pins of conical profile will be optimal for the combination of high conduction coefficient and fin thickness or for low convection coefficient of the cooling fluid.

Conclusions

A differential transform method to analyze stationary heat conduction through homogeneous extended surfaces with negligible heat loss from the tip has been presented. The major conclusion of this work is that for stationary heat conduction through extended surfaces, the DTM solution can be obtained in a closed series solution which does not necessitate an iterative method such as the Newton–Raphson methods for the determination of the initial value of temperature transform function. The proposed method is shown to converge with a few Taylor series for both low and high values of the thermal length characteristic parameter. Application of the present DTM solution to longitudinal fins of rectangular and triangular profiles and pins of cylindrical and conical profiles show an excellent agreement with exact results. For all cases studied, the magnitude of temperature decreases with increasing thermal length characteristic parameter. This indicates that the loss of heat from extended surfaces is more significant for low convection coefficient of the cooling fluid than that for extended surfaces with high thickness and conduction coefficient.

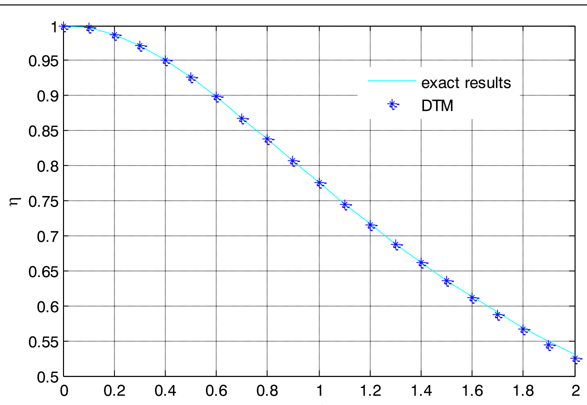

Fig. 9 Efficiency for a conical pin

Nomenclature

A fin section area, m^2

b height of the fin, m

h heat convectivity coefficient $W \cdot m^{-2} \cdot K^{-1}$

k_c thermal conductivity heat coefficient, $W \cdot m^{-1} \cdot K^{-1}$

q heat flux, W/m^2

T temperature, K

x coordinate, m

p profile function

P profile transform function

W thickness of the fin, m

Greek symbols

β dimensionless temperature

θ dimensionless temperature

Θ dimensionless temperature transform function

η fin efficiency

ζ dimensionless coordinate

Subscripts

b fin base

i ideal parameter

∞ ambient environment

Authors' contributions

ELL implemented the quasi-analytic differential transform method and HTTK defined the research problem. All authors contributed to the problem formulation, drafted the manuscript, and read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

Received: 1 July 2017 Accepted: 13 July 2017
Published online: 16 August 2017

References

Arauzo, I., Campo, A., & Cortes, C. (2005). Quick estimate of the heat transfer characteristics of annular fins of hyperbolic profile with the power series method. *Applied Thermal Engineering, 25*, 623–634.

Arslanturk, C. (2005). A decomposition method for fin efficiency of convective straight fins with temperature-dependent thermal conductivity. *International Communications in Heat and Mass Transfer, 32*, 831–841.

Aziz, A., & Bouaziz, M. N. (2011). A least squares method for a longitudinal fin with temperature dependent internal heat generation and thermal conductivity. *Energy Conversion and Management, 52*, 2876–2882.

Bhowmik, A., Singla, R. K., Roy, P. K., Prasad, D. K., Das, R., & Repaka, R. (2013). Predicting geometry of rectangular and hyperbolic fin profiles with

temperature-dependent thermal properties using decomposition and evolutionary methods. *Energy Conversion and Management, 74*, 535–547.
Brestovič, T., Jasminská, N., & Lázár, M. (2015). Application of analytical solution for extended surfaces on curved and squared ribs. *Acta Mechanica et Automatica, 9*(2), 75–83.
Chen, C. K., & Ho, S. H. (1999). Solving partial differential equations by two-dimensional differential transform method. *Applied Mathematics and Computation, 106*, 171–179.
Chen, C. K., & Ju, S. P. (2004). Application of differential transformation to transient advective–dispersive transport equation. *Applied Mathematics and Computation, 155*, 25–38.
Diez, L. I., Campo, A., & Cortés, C. (2009). Quick design of truncated pin fins of hyperbolic profile for heat-sink applications by using shortened power series. *Applied Thermal Engineering, 29*, 815–821.
Domairry, G., & Fazeli, M. (2009). Homotopy analysis method to determine the fin efficiency of convective straight fins with temperature-dependent thermal conductivity. *Communications in Nonlinear Science and Numerical Simulation, 14*, 489–499.
Hassan, I. H. A. (2002). Different applications for the differential transformation in the differential equations. *Applied Mathematics and Computation, 129*, 183–201.
Hayat, T., Waqas, M., Ijaz Khan, M., & Alsaedi, A. (2016). Analysis of thixotropic nanomaterial in a doubly stratified medium considering magnetic field effects. *International Journal of Heat and Mass Transfer, 102*, 1123–1129.
Hayat, T., Ijaz Khan, M., Waqas, M., & Alsaedi, A. (2017a). Newtonian heating effect in nanofluid flow by a permeable cylinder. *Results in Physics, 7*, 256–262.
Hayat, T., Ijaz Khan, M., Farooq, M., Alsaedi, A., & Khand, M. I. (2017b). Thermally stratified stretching flow with Cattaneo-Christov heat flux. *International Journal of Heat and Mass Transfer, 106*, 289–294.
Inc, M. (2008). Application of homotopy analysis method for fin efficiency of convective straight fins with temperature-dependent thermal conductivity. *Mathematics and Computers in Simulation, 79*, 189–200.
Jang, M.-J., Yeh, Y.-L., Chen, C.-L., & Yeh, W.-C. (2010). Differential transformation approach to thermal conductive problems with discontinuous boundary condition. *Applied Mathematics and Computation, 216*, 2339–2350.
Joneidi, A. A., Ganji, D. D., & Babaelahi, M. (2009). Differential transformation method to determine fin efficiency of convective straight fins with temperature dependent thermal conductivity. *International Communications in Heat and Mass Transfer, 36*, 757–762.
Kang, H.-S. (2009). Optimization of a rectangular profile annular fin based on fixed fin height. *Journal of Mechanical Science and Technology, 23*, 3124–3131.
Kraus, A. D., Aziz, A., & Welty, J. (2001). *Extended surface heat transfer*. Hoboken: John Wiley & Sons, Inc.
Kundu, B., & Lee, K.-S. (2012). Analytic solution for heat transfer of wet fins on account of all nonlinearity effects. *Energy, 41*, 354e367.
Roy, P. K., & Mallick, A. (2016). Thermal analysis of straight rectangular fin using homotopy perturbation method. *Alexandria Engineering Journal, 55*, 2269–2277.
Torabi, M., & Zhang, Q. B. (2013). Analytical solution for evaluating the thermal performance and efficiency of convective–radiative straight fins with various profiles and considering all non-linearities. *Energy Conversion and Management, 66*, 199–210.
Waqas, M., Farooq, M., Khan, M. I., Ahmed Alsaedi, A., Hayat, T., & Yasmeen, T. (2016). Magnetohydrodynamic (MHD) mixed convection flow of micropolar liquid due to nonlinear stretched sheet with convective condition. *International Journal of Heat and Mass Transfer, 102*, 766–772.
Yaghoobi, H., & Torabi, M. (2011). The application of differential transformation method to nonlinear equation arising in heat transfer. *International Communications in Heat and Mass Transfer, 38*, 815–820.